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SOLUTION OF THE AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR TRANSVERSELY ISOTROPIC BODIES WITH THE AID OF GENERALIZED ANALYTIC FUNCTIONS

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The solution of the axisymmetric problem for transversely isotropic bodies, expressed in terms of generalized analytic functions, is constructed. We obtain formulas for the displacements and stresses, similar to the corresponding formulas of the plane problem. The representation of the generalized analytic functions by analytic ones, are indicated, and the analogue of the Cauchy-type integral which gives the possibility of reducing the boundary value problems to integral equations, is presented. As an example, we consider the action of forces which are distributed along a circumference in the interior of a transversely isotropic space.

The plane problems of the theory of elasticity for transversely isotropic bodies are solved effectively with the aid of analytic functions of a complex variable [1]. In [2, 3] the solution of axisymmetric and nonaxisymmetric problems for bodies of revolution with the aid of analytic functions and contour integrals, was considered. In the case of an isotropic elastic medium, the solution of axisymmetric problems with the aid of a class of generalized analytic functions [4] was proposed.

1. Let $U_k(z, r)$ and $V_k(z, r)$ be complex functions satisfying the system of equations

$$\gamma_k \frac{\partial U_k}{\partial z} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) V_k, \qquad \frac{\partial U_k}{\partial r} = -\gamma_k \frac{\partial V_k}{\partial z} \qquad (k = 1; 2) \tag{1.1}$$

where the parameter γ_k is some number, in general, complex. These functions, obviously, satisfy the differential equations

$$\Delta_k U_k = 0, \qquad \left(\Delta_k - \frac{1}{r^2}\right) V_k = 0 \tag{1.2}$$

$$\left(\Delta_k = \gamma_k^2 \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right)$$

We introduce the complex variables $t_k = z_k + ir$ and $t_k^* = z_k - ir$ $(z_k = z / \gamma_k)$ and the functions

$$\Phi_k(t_k, t_k^*) = U_k(z, r) + iV_k(z, r) \tag{1.3}$$
$$\Phi_k^*(t_k, t_k^*) = U_k(z, r) - iV_k(z, r)$$

Then, the system (1.1) can be written in the form

$$2\frac{\partial \Phi_k}{\partial t_k^*} - \frac{\Phi_k - \Phi_k^*}{t_k - t_k^*} = 0 \qquad \left(2\frac{\partial}{\partial t_k^*} = \frac{\partial}{\partial z_k} + i\frac{\partial}{\partial r}\right) \tag{1.4}$$
$$2\frac{\partial \Phi_k^*}{\partial t_k} - \frac{\Phi_k - \Phi_k^*}{t_k - t_k^*} = 0 \qquad \left(2\frac{\partial}{\partial t_k} = \frac{\partial}{\partial z_k} - i\frac{\partial}{\partial r}\right)$$

On the above introduced functions we impose parity conditions, considering U_k an even function with respect to r and V_k an odd function. Then

$$U_k(z, r) = U_k(z, -r), \qquad V_k(z, r) = -V_k(z, -r) \tag{1.5}$$
$$\Phi_k(t_k, t_k^*) = \Phi_k^*(t_k^*, t_k)$$

In the sequel it will be necessary to consider two cases for the value of the parameter γ_k $(k = 1, 2)$: case (a), when γ_1 and γ_2 are complex conjugate numbers $(\gamma_k = \bar{\gamma}_j;\ k + j = 3)$ and case (b), when γ_1 and γ_2 are real distinct numbers $(\gamma_k = \bar{\gamma}_k,\ \gamma_k \neq \gamma_j)$.

In the case (a) $t_k^* = \bar{t}_j$ and it is possible, without violating the previous equalities, to introduce the relations

$$U_k = \bar{U}_j, \qquad V_k = \bar{V}_j, \qquad \Phi_k^*(t_k, t_k^*) = \overline{\Phi_j(t_j, t_j^*)} \tag{1.6}$$

In the case (b) $t_k^* = t_k$, and we can consider that the functions U_k and V_k are real. Then

$$U_k = \bar{U}_k, \qquad V_k = \bar{V}_k, \qquad \Phi_k^*(t_k, t_k^*) = \overline{\Phi_k(t_k, t_k^*)} \tag{1.7}$$

Substituting (1.6) and (1.7) into (1.4) we can see that in the case (b) $\Phi_k(t_k, \bar{t}_k)$ is a generalized analytic function in the sense of Vekua [5]. In the case (a) they belong to a somewhat larger class, nevertheless, the basic properties of generalized analytic functions hold for them also.

2. We consider the axisymmetric problem of the theory of elasticity for a transversely isotropic medium. The differential equations of the equilibrium in the cylindrical coordinates z, r (z is the symmetry axis) have the form

$$\left[A_{11}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + A_{44}\frac{\partial^2}{\partial z^2}\right]u + (A_{13} + A_{44})\frac{\partial^2 w}{\partial z \partial r} = 0 \tag{2.1}$$
$$(A_{13} + A_{44})\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial u}{\partial z} + \left[A_{44}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial}{\partial r} + A_{33}\frac{\partial^2}{\partial z^2}\right]w = 0$$

Here w and u are the axial and the radial displacements, $A_{11}, A_{13}, A_{33}, A_{44}$ are the moduli of elasticity [1].

We denote

$$A_k = (A_{33} - A_{44}\gamma_k^2) / (A_{11}A_{44}) \tag{2.2}$$
$$B_k = (A_{11}\gamma_k^2 - A_{44}) / (A_{11}A_{44})$$
$$D = (A_{13} + A_{44}) / (A_{11}A_{44}), \qquad (D^2\gamma_k^2 = A_kB_k)$$

Here by γ_k $(k = 1; 2)$ we understand the roots of the characteristic equation (given between parentheses in (2.2)), satisfying the condition $\operatorname{Re}\gamma_k > 0$. These roots are the inverses of the parameter s_k, introduced in [1] for the case of the axisymmetric deformation and can be either complex conjugates or real; the case of equal roots will not be considered here.

We introduce the functions F and f by the relations

$$A_k \frac{\partial w}{\partial z} + D\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) u = F, \qquad B_k \frac{\partial u}{\partial z} - D\frac{\partial w}{\partial r} = \frac{f}{A_{44}} \tag{2.3}$$

Then Eqs. (2.1) can be written in the form

$$A_{33} D \frac{\partial F}{\partial z} = A_k \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) f, \qquad A_{11} B_k \frac{\partial F}{\partial r} = -D\frac{\partial f}{\partial z} \tag{2.4}$$

Let us associate (2.4) with (1.1). Taking into account the equality $A_{33} / A_{11} = \gamma_k^2\gamma_j^2$ $(k + j = 3)$, we can see that the functions $A_{11}F$ and αf, where $\alpha = D / (B_k\gamma_j)$, satisfy the system (1.1) with the parameter γ_j; their combination of type (1.3) forms a generalized analytic function.

We will consider the equalities (2.3) as a system of differential equations relative to w and u. The general solution of this system has the form

$$w = p_k U_k + p_j U_j, \qquad u = -q_k V_k - q_j V_j \tag{2.5}$$
$$(q_k = p_k \gamma_k D / B_k)$$

Here the first terms represent the general solution of the system (2.3) with right-hand sides equal to zero; the functions U_k and V_k satisfy Eqs. (1.1) with the parameter γ_k. The second terms represent particular solutions of the complete system (2.3). We will assume that U_j and V_j are related to the right-hand sides of this system by the equalities

$$p_j \frac{\partial U_j}{\partial z} = \frac{A_{11} F}{\gamma_j^2 - \gamma_k^2}, \qquad p_j \frac{\partial V_j}{\partial z} = \frac{\alpha f}{\gamma_j^2 - \gamma_k^2}$$

and satisfy Eqs. (1.1) with the parameter γ_j (which is possible by virtue of (2.4)). The factors p_k and p_j are arbitrary, except that they are either complex conjugate or real numbers. Their values will be taken from considerations of convenience in the writing of the stress formulas.

Now we turn to the generalized analytic functions with the aid of (1.3). Taking into account (1.6), (1.7), we can see that for the cases (a) and (b) the formulas (2.5) can be written in the form

$$w = \operatorname{Re}(p_1\Phi_1 + p_2\Phi_2), \qquad u = \operatorname{Re}(iq_1\Phi_1 + iq_2\Phi_2) \tag{2.6}$$

If z, r are considered as rectangular coordinates in the meridian plane of the body, then w is an even function with respect to r, while u is an odd function. This is in agreement with the conditions (1.5).

3. For the determination of stresses it is convenient to switch to the technical elastic constants. The characteristic equation and the coefficients of the formulas (2.5), (2.6) will have the form

$$\left(1 - \nu_z^2 \frac{E_r}{E_z}\right)\gamma_k^4 - \left[\frac{E_z}{G_z} - 2\nu_z(1 + \nu_r)\right]\gamma_k^2 + (1 - \nu_r^2)\frac{E_z}{E_r} = 0$$

$$p_k = -\frac{\gamma_k}{E_z}\left[\left(1 - \nu_z^2 \frac{E_r}{E_z}\right)\gamma_k^2 + \nu_z(1 + \nu_r)\right]$$

$$q_k = -\frac{1}{E_z}\left[(1-\nu_r^2)\frac{E_z}{E_r} + \nu_z(1+\nu_r)\gamma_k^2\right]$$

Making use of the known relations which connect the displacements and the stresses of a transversely isotropic elastic body, we find the expressions for the stresses in terms of the generalized analytic functions

$$\begin{aligned}
\sigma_z &= -\operatorname{Re}(\gamma_1^2\Phi_1' + \gamma_2^2\Phi_2') \qquad \left(\Phi_k' = \gamma_k\frac{\partial}{\partial z}\Phi_k\right) \qquad (3.1)\\
\tau_{zr} &= -\operatorname{Re}(i\gamma_1\Phi_1' + i\gamma_2\Phi_2')\\
\sigma_r &= \operatorname{Re}(\Phi_1' + \Phi_2') - 2G_r\frac{u}{r}\\
\sigma_\theta &= \nu_r\sigma_r + \nu_z\frac{E_r}{E_z}\sigma_z + E_r\frac{u}{r} \qquad (r \neq 0)\\
\sigma_r &= \sigma_\theta = \frac{1+\nu_r}{2}\operatorname{Re}(\Phi_1' + \Phi_2') + \frac{\nu_z}{2}\frac{E_r}{E_z}\sigma_z \qquad (r = 0)
\end{aligned}$$

The formulas (2.6) and (3.1) are similar to the corresponding formulas which express the components of the plane strain in terms of analytic functions [1].

For specified stresses, the functions Φ_k are determined except for an expression $a_k + ib_k / r$ (generalized constant), where

$$q_1 b_1 + q_2 b_2 = 0 \qquad (a_1 = \bar{a}_2,\ b_1 = \bar{b}_2 \quad \text{or} \quad \operatorname{Im} a_k = \operatorname{Im} b_k = 0)$$

If the displacements are specified, then one has to put in addition $p_1 a_1 + p_2 a_2 = 0$.

4. The Eqs. (1.1) and the equalities (1.5) are satisfied if the functions U_k and V_k are represented in the form of the integrals

$$U_k(z, r) = \frac{r}{\pi i\,|r|}\int_{t_k^*}^{t_k}\frac{\varphi_k(\zeta_k)\,d\zeta_k}{\sqrt{(\zeta_k - t_k)(\zeta_k - t_k^*)}} \qquad (4.1)$$

$$V_k(z, r) = -\frac{1}{\pi i\,|r|}\int_{t_k^*}^{t_k}\frac{\varphi_k(\zeta_k)(\zeta_k - z_k)\,d\zeta_k}{\sqrt{(\zeta_k - t_k)(\zeta_k - t_k^*)}}$$

Here $\varphi_k(\zeta_k)$ is an analytic function of the complex variable $\zeta_k = x_k + iy$ $(x_k = x / \gamma_k)$. From here we obtain the representation of the generalized analytic functions in terms of the analytic ones

$$\Phi_k(t_k, t_k^*) = -\frac{2}{|t_k - t_k^*|}\int_{t_k^*}^{t_k}\ {}_k(\zeta_k)\sqrt{\frac{\zeta_k - t_k^*}{\zeta_k - t_k}}\,d\zeta_k \qquad (4.2)$$

$$\Phi_k' = \frac{\partial}{\partial z_k}\Phi_k = -\frac{2}{|t_k - t_k^*|}\int_{t_k^*}^{t_k}\frac{d\varphi_k}{d\zeta_k}\sqrt{\frac{\zeta_k - t_k^*}{\zeta_k - t_k}}\,d\zeta_k \qquad (4.3)$$

The conditions (1.6) and (1.7) are satisfied if we set, respectively,

$$\varphi_k(\zeta_k) = \overline{\varphi_j(\bar{\zeta}_j)}, \qquad \varphi_k(\zeta_k) = \overline{\varphi_k(\bar{\zeta}_k)} \qquad (4.4)$$

The expressions (4.1) – (4.3) represent the natural generalizations of the results obtained in [4]. They are suitable when the plane domain, occupaied by the meridian plane of the body, intersects the z-axis. In this case, for the points of the z-axis the equalities (4.1), (4.2) become

$$\Phi_k(z_k, z_k) = U_k(z, 0) = \varphi_k(z_k), \qquad V_k(z, 0) = 0 \tag{4.5}$$

We note that in the formulas (4.1) – (4.3) we can pass to integration along the contour of the domain (introducing the factor $^1/_2$) and substitute the boundary values of the analytic function by an arbitrary function defined on the contour (by the density of the integral) [4]. All these representations can be used in order to express with the aid of (3.1) and (2.6) the stress and displacement components in terms of analytic functions or contour integrals. Similar expressions (for the case (b)) have been obtained by a different method in [2] (see also [3]).

5. For the generalized analytic functions we have the analogue of the Cauchy type integral

$$\Phi_k(t_k, t_k^*) = \frac{1}{2\pi i}\int_L F_k(\tau_k) W(t_k, \tau_k)\, d\tau_k \tag{5.1}$$

$$W = \frac{\omega}{\tau_k - t_k}, \qquad \omega = [E(\mu_k) - (1-\mu_k^2) K(\mu_k)] \frac{|\tau_k - \tau_k^*|}{\rho_k \mu_k^2} \tag{5.2}$$

$$\mu_k = \frac{1}{\rho_k}\sqrt{(\tau_k - \tau_k^*)(t_k^* - t_k)}, \qquad \rho_k = \sqrt{(\tau_k - t_k^*)(\tau_k^* - t_k)}$$

Here $W(t_k, \tau_k)$ is the generalized Cauchy kernel, K and E are the complete elliptic integrals, $F_k(\tau_k)$ is the density of the integral satisfying the conditions (4.4), L is the contour of the plane domain occupied by the meridian section of the body, τ_k is the affix of a contour point. These formulas can be easily obtained by the method applied in [4].

The representation (5.1) can be used for the reduction of the boundary problems to integral equations, corresponding to Sherman's equations for the corresponding plane problem [6].

6. As an example we consider the action of forces, distributed along the circumference $(r = r_0,\ z = z_0)$ inside a transversely isotropic space and directed along the z-axis. By arguments similar to those given in [7], the following representation for the generalized analytic functions was obtained:

$$\Phi_k = P r_0 N_k \left\{ \frac{1}{\rho_k} K + \frac{i}{r}\left[\frac{z_k - z_{0k}}{2\rho_k}(n'\Pi + K) + \frac{\pi}{2}(A - 1)\right]\right\}, \qquad N_k = \frac{p_k \gamma_j^2}{\pi\gamma_k(\gamma_k^2 - \gamma_j^2)}\,\frac{E_r}{1-\nu_r^2} \tag{6.1}$$

$$n' = (r - r_0)/(r + r_0), \qquad n^2 = 1 - n'^2, \qquad z_{0k} = z_0/\gamma_k$$

Here $K = K(\mu_k)$, $\Pi = \Pi(-n^2, \mu_k)$ are the complete elliptic integrals of the first and the third kind, respectively, A is a piecewise-constant function described in [7] and P is the intensity of the distributed forces.

The corresponding formulas for the displacements and stresses follow from (2.6) and (3.1).

If we set $2\pi r_0 P = P_0 = \mathrm{const}$ and let r_0 tend to zero, then from (6.1) for $z_0 = 0$ we obtain

$$\Phi_k = \frac{1}{4} P_0 N_k \left(\frac{r + iz_k}{r\sqrt{z_k^2 + r^2}} - \frac{i}{r} \right)$$

$$w = \frac{1}{4} P_0 \sum_{k=1}^{2} \frac{N_k p_k}{\sqrt{z_k^2 + r^2}}$$

$$u = -\frac{1}{4r} P_0 \sum_{k=1}^{2} N_k q_k \left(\frac{z_k}{\sqrt{z_k^2 + r^2}} - 1 \right)$$

The last formulas coincide essentially with the known solution of the action of a concentrated force P_0 inside a transversely isotropic space (see, for example, [8]).

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